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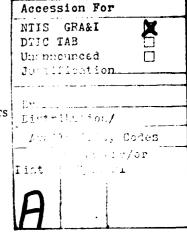1 JAN 73

RECENT RESULTS CONCERNING THE EFFECTS OF NONLINEARITIES ON RANDOM INPUTS

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Abstract

This paper surveys some recent results concerning the effects of a zero memory nonlinearity on stationary random processes. Emphasis is placed upon the mean square continuity of the output, the autocorrelation function of the output, equivalence classes of zero memory nonlinearities, and bandwidth properties of the output. A new result concerning the effect of a zero memory nonlinearity on the spectrum of randomly modulated Gaussian noise is presented.

Introduction

From the viewpoint of second moment theory, the study of linear systems with random inputs has been fairly well developed. However, such is not the case for nonlinear systems. The difficulties that arise in problems associated with nonlinear systems with random inputs are frequently insurmountable, and a completely general approach to nonlinear transformations upon random processes does not appear possible. Much of the success in this area has been achieved by restricting attention to certain situations involving particular classes of random inputs and particular forms of nonlinear systems.

One form of nonlinear system that has been successfully analyzed to a good degree is the zero memory nonlinearity (ZNL). In statistical communication theory, as well as other areas of engineering, we are frequently interested in ZNL's with random inputs. For example, limiters, rectifiers, power law devices, and other ZNL's are an integral part of many systems.

In this paper we will be concerned with time invariant ZNL's. A ZNL will denote a transformation such that the output at time t depends only on the input at time t. The input will be denoted by $X(t)$. In the sequel we will assume that $X(t)$ is a zero mean, second order random process that is second order stationary and mean square continuous. The parameter set, denoting the time index, will be the real line. The general class of ZNL's $\mathscr{G}$ will be taken as the class of all Borel measurable functions $g(\cdot)$ such that $g[X(t)]$ is a second order random process. We will identify any two ZNL's $g_1(\cdot)$ and $g_2(\cdot)$ such that $g_1[X(t)]$ and $g_2[X(t)]$ are equivalent random processes. The output of the ZNL will be denoted by $Y(t)$. Thus $Y(t)$ is a second order random process that is second order stationary.

Mean Square Continuity

In the analysis of second order random processes, the property of mean square continuity is frequently assumed. We recall that the mean square continuity of $Y(t)$ is given by the condition that

$$\lim_{s \to t} E\left\{[Y(s) - Y(t)]^2\right\} = 0$$

for all t. For $Y(t)$, mean square continuity is a necessary and sufficient condition for the existence of a measurable standard modification of the random process[1-3]. Measurability of a random process is necessary to take full advantage of the apparatus of measure theory. For example, measurability is used to justify exchanging the order of the expectation operator and the integral of a random process[4]. In applications, this is pertinent to the linear filtering of $Y(t)$. Also, mean square continuity is a necessary and sufficient condition[5] for the popular spectral representation of $Y(t)$. In some general treatments[2,6,7] of wide sense stationary random processes, mean square continuity is taken as a standing assumption.
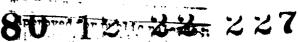
The following result is due to Wise and Thomas[8].

Theorem 1: If $g \in \mathscr{G}$, then $Y(t) = g[X(t)]$ is mean square continuous.

On the basis of this result, we see that the output of the ZNL is mean square continuous, even though the ZNL $g(\cdot)$ might not be continuous. Thus, from the preceding comments, we see that $Y(t)$ possesses a spectral representation. It is the spectral representation which affords us the interpretation of the bandwidth properties of $Y(t)$, which will be considered later in the paper. In summary then, we see that the output $Y(t)$ is a second order random process that is second order stationary, that is mean square continuous, that possesses a spectral representation, and that is equivalent to a measurable random process.

The Autocorrelation Function of the Output

In this section we will assume that X(t) is a Gaussian process with a positive variance σ^2 and a normalized autocorrelation function $\rho(\tau)$. Let $R_Y(\tau)$ denote the autocorrelation function of the output

$$R_Y(\tau) = E\left\{Y(t)\ Y(t+\tau)\right\}$$
$$= E\left\{g[X(t)]\ g[X(t+\tau)]\right\} . \tag{1}$$

A direct approach to evaluate $R_Y(\tau)$ is often difficult. Instead, several indirect approaches have been introduced which frequently simplify the evaluation of $R_Y(\tau)$. These include Price's Theorem[9-13], transform methods, and series expansions. A collection of useful techniques for studying the output autocorrelation function of a ZNL with a Gaussian input is presented by Thomas[14]. In particular, limiters and power law devices are treated, and many useful references are given.

We will outline a general method for analyzing $R_Y(\tau)$ which is based upon a method introduced by Barrett and Lampard[15]. Let

$$\theta_n(x) = \frac{H_n\left(\frac{x}{\sigma}\right)}{\sqrt{n!}} , \tag{2}$$

where $H_n(x)$ is the n-th Hermite polynomial defined by

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right) ,$$

and let

$$b_n = \frac{1}{\sqrt{2\pi}\ \sigma} \int_{-\infty}^{\infty} \theta_n(x) g(x) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$
$$\tag{3}$$
$$= E\left\{\theta_n[X(t)]\ g[X(t)]\right\} .$$

Then the output autocorrelation function is given by

$$R_Y(\tau) = \sum_{n=0}^{\infty} \left(b_n\right)^2 [\rho(\tau)]^n . \tag{4}$$

This series is absolutely and uniformly convergent. If the sum is truncated, the result is still a valid autocorrelation function; that is, it is nonnegative definite. Also, the truncation error can be uniformly bounded in a straightforward manner, since $|\rho(\tau)| \leq 1$. If the ZNL $g(\cdot)$ is a polynomial then the coefficient sequence will truncate, and the coefficients will be easy to obtain.

Consider attempting to identify the ZNL $g(\cdot)$ from a knowledge of $R_Y(\tau)$ and the input autocorre-

lation function $\sigma^2\rho(\tau)$. Although it may not be clear from (1), it can be seen from (4) that this is, in general, not possible. That is, we see from (3) that

$$g(x) = \sum_{n=0}^{\infty} b_n\theta_n(x) \tag{5}$$

where the series is mean square convergent. Thus we see that the ZNL $g(\cdot)$ is determined by the sequence $\{b_n\}$, while from (4) we see that the output autocorrelation function is determined by the sequence $\left\{(b_n)^2\right\}$. This motivates the concept of equivalence classes[16,17] of ZNL's. That is, the equivalence class of a ZNL is the set of all ZNL's which produce exactly the same transformation upon the autocorrelation function of the input. The equivalence class of a ZNL is therefore the set of all ZNL's that can be obtained by changing the signs of the coefficients in (5). It is straightforward to see that this does result in an equivalence relation; that is, it is reflexive, symmetric, and transitive. We note that the sum in (5) truncates if and only if $g(\cdot)$ is a polynomial. Thus, the equivalence class of a ZNL is a finite set if and only if the ZNL is a polynomial.

Utilizing the equivalence class, we can give examples of ZNL's having greatly different structural properties which produce exactly the same output autocorrelation function when the input is given by X(t). For example, the following three ZNL's belong to the same equivalence class[17]:

$$g_1(x) = \sin\left(\frac{x\sqrt{2}}{\sigma}\right)$$

$$g_2(x) = \sin\left(\frac{x\sqrt{2}}{\sigma}\right) - \left(\frac{2\sqrt{2}}{e\sigma}\right)x$$

$$g_3(x) = e^{-2} \sinh\left(\frac{x\sqrt{2}}{\sigma}\right) .$$

By utilizing the equivalence class, one can preserve the second moment properties associated with a given ZNL but perhaps choose a ZNL having more acceptable functional properties, e.g. dynamic range, ease of implementation, etc.

Bandwidth Properties of the Output

As we noted earlier, the output Y(t) will possess a spectral representation. In this section we will consider how the ZNL affects the bandwidth properties of X(t). We will assume throughout the sequel that the ZNL $g(\cdot)$ is not equivalent to a constant function; otherwise, the spectral distribution function of Y(t) would simply be a step function.

We will begin by reviewing some results due to Abramson[18]. Assume that the input random process X(t) possesses a mean square derivative, denoted by

$\dot{X}(t)$. We recall[19] that $X(t)$ possesses a mean square derivative if and only if the autocorrelation function $R(\tau)$ of $X(t)$ possesses a second derivative at the origin. Let BW_X denote the second moment bandwidth of $X(t)$. That is, BW_X^2 is the second moment of the normalized spectral distribution of $X(t)$, given by

$$BW_X^2 = \frac{\int_{-\infty}^{\infty} \omega^2 \, dF(\omega)}{\int_{-\infty}^{\infty} dF(\omega)}$$

$$= - \frac{R''(0)}{R(0)} \, ,$$

where $F(\cdot)$ denotes the spectral distribution function of $X(t)$. Similarly, let BW_Y denote the second moment bandwidth of $Y(t)$, and let $F_Y(\cdot)$ denote the spectral distribution function of $Y(t)$. Then

$$BW_Y^2 = \frac{\int_{-\infty}^{\infty} \omega^2 \, dF_Y(\omega)}{\int_{-\infty}^{\infty} dF_Y(\omega)} \, .$$

We note[20] that BW_Y is finite if and only if $R_Y(\tau)$ possesses a second derivative at the origin. If BW_Y is finite, we have

$$BW_Y^2 = - \frac{R_Y''(0)}{R_Y(0)} \, .$$

We will say that the input $X(t)$ is derivative independent if the two random variables $X(t)$ and $\dot{X}(t)$ are independent. (Note that we are <u>not</u> assuming that the two random processes $X(t)$ and $\dot{X}(t)$ are independent.) The following result is due to Abramson[18].

<u>Theorem 2:</u> If $g \in \mathscr{G}$ is absolutely continuous, if $h(x)=g'(x)$, and if $X(t)$ is a derivative independent random process, then the second moment bandwidth of the output BW_Y is related to the second moment bandwidth of the input BW_X via

$$BW_Y^2 = \frac{E\left\{(h[X(t)])^2\right\} \cdot R(0)}{E\left\{(g[X(t)])^2\right\}} \, BW_X^2 \, .$$

Notice that if $X(t)$ is a mean square differentiable Gaussian random process, then it is derivative independent. That is, since $R(\tau)$ is twice differentiable and is maximized at the origin, then

$$R'(0) = 0$$

$$= E\left\{X(t) \, \dot{X}(t)\right\} \, .$$

Since $X(t)$ and $\dot{X}(t)$ are mutually Gaussian and uncorrelated, they are independent. For example, for

$$g(x) = x^3,$$

and

$$E\left\{[X(t)]^2\right\} = 1 \, ,$$

we have

$$BW_Y = \sqrt{1.8} \, BW_X .$$

The following result, due to Wise, Traganitis, and Thomas[16], gives a general relation between BW_X and BW_Y when $X(t)$ is a Gaussian process.

<u>Theorem 3:</u> Suppose that $X(t)$ is a zero mean, stationary Gaussian process that has a finite second moment bandwidth BW_X and that possesses a spectral density function. If $g \in \mathscr{G}$ is nonconstant and is such that $E\left\{g[X(t)]\right\} = 0$, then the second moment bandwidth of $Y(t) = g[X(t)]$ is greater than or equal to BW_X. Equality holds if and only if $g(\cdot)$ is linear.

As a corollary to Theorem 3, it also follows that

$$\left[\frac{BW_Y}{BW_X}\right]^2 = \frac{\sum_{n=1}^{\infty} n\left(b_n\right)^2}{\sum_{n=1}^{\infty} \left(b_n\right)^2} \, ,$$

where b_n is given by (3).

It follows from Theorem 2 and Theorem 3 that in these respective cases, the relation between BW_X and BW_Y does not depend upon the shape of the input spectral distribution. Instead, it depends only upon the ZNL and the first order distribution of the input process. Some examples where Theorems 2 and 3 hold with non-Gaussian inputs are given in the respective papers.

Now we will consider the strict bandlimitedness of the output $Y(t)$. A random process is strictly bandlimited if the spectral distribution has bounded support. The following result, due to the author[21], completely characterizes the strict bandlimitedness of the output when the input is a Gaussian process.

<u>Theorem 4:</u> Let $X(t)$ be a stationary, mean square continuous Gaussian random process with a nonconstant autocorrelation function, and let $g \in \mathscr{G}$. Then $g[X(t)]$ is strictly bandlimited if and only if

a) $X(t)$ is strictly bandlimited, and

b) $g(\cdot)$ is a polynomial.

Notice that many common ZNL's are not polynomials. In particular, it follows that if $X(t)$, given in Theorem 4, is passed through any type of limiter, then the output cannot be strictly bandlimited.

The following result[21] establishes necessary conditions for the output to be strictly bandlimited when the input is a "contaminated Gaussian process".

Theorem 5: Let $G(t)$ and $A(t)$ be mutually independent, mean square continuous, second order stationary random processes. Assume that $G(t)$ is a zero mean Gaussian process with a nonconstant autocorrelation function. Let $X(t) = G(t) + A(t)$, and let $g \in \mathcal{G}$.

   A. If $g(\cdot)$ is not a polynomial, then $g[X(t)]$ cannot be strictly bandlimited for any $A(t)$.

   B. If $G(t)$ is not strictly bandlimited, then $g[X(t)]$ cannot be strictly bandlimited for any nonconstant $g(\cdot)$.

In the sequel we will consider "randomly modulated Gaussian processes". To be specific, let $G(t)$ and $A(t)$ be as given in Theorem 5, and let $X(t) = A(t) G(t)$. If ξ and η are zero mean mutually Gaussian random variables with correlation coefficient α, and if g_1 and g_2 are two Borel measurable functions such that $g_1(\xi)$ and $g_2(\eta)$ are both second order, then it follows[22] that

$$E\left\{g_1(\xi)\ g_2(\eta)\right\} = \sum_{n=0}^{\infty} \alpha^n\ b_n^{(1)} b_n^{(2)}$$

where

$$b_n^{(1)} = E\left\{g_1(\xi)\ \theta_n(\xi)\right\}$$

and

$$b_n^{(2)} = E\left\{g_2(\eta)\ \theta_n(\eta)\right\}$$

and $\theta_n(\cdot)$ is given by (2). Using this result we have that the following conditional expectation is almost surely given by

$$E\left\{g[X(t)]\ g[X(s)]\ |\ A(t),\ A(s)\right\} =$$

$$\sum_{n=0}^{\infty} [\rho(t-s)]^n\ b_n[A(t)] b_n[A(s)]$$

where

$$b_n(y) = E\left\{g[yG(t)]\ \theta_n[G(t)]\right\}$$

and where ρ is the normalized autocorrelation function of $G(t)$. Thus we see that the output autocorrelation function is given by

$$E\left\{g[X(t)]\ g[X(s)]\right\} =$$

$$E\left\{\sum_{n=0}^{\infty} [\rho(t-s)]^n\ b_n[\ A(t)]\ b_n[A(s)]\right\}.$$

Notice that

$$\left|[\rho(t-s)]^n\ b_n[A(t)]\ b_n[A(s)]\ \right| \leq$$

$$\left(b_n[A(t)]\right)^2 + \left(b_n[A(s)]\right)^2$$

and

$$E\left\{\sum_{n=0}^{\infty}\left[\left(b_n[A(t)]\right)^2 + \left(b_n[A(s)]\right)^2\right]\right\} =$$

$$2\ E\left\{\left(g[X(t)]\right)^2\right\} < \infty\ .$$

Thus the Dominated Convergence Theorem justifies exchanging the expectation and the infinite summation. Therefore, we have that

$$E\left\{g[X(t)]\ g[X(s)]\right\} =$$

$$\sum_{n=0}^{\infty} [\rho(t-s)]^n\ E\left\{b_n[A(t)]\ b_n[A(s)]\right\}.$$

Notice that $b_n(\cdot)$ can be interpreted as a ZNL and thus $b_n[A(t)]$ is a second order random process that is second order stationary. Thus $E\left\{b_n[A(t)]\ b_n[A(s)]\right\}$ is a nonnegative definite function of $(t-s)$, say $R_n(t-s)$, and it follows from Theorem 1 that $R_n(\cdot)$ is continuous. Therefore,

$$E\left\{g[X(t)]\ g[X(s)]\right\} =$$

$$\sum_{n=0}^{\infty} [\rho(t-s)]^n\ R_n(t-s). \tag{6}$$

The author has shown in another paper[21] that if the output autocorrelation function has the form of (6) where $\rho(\cdot)$ is nonconstant and $R_n(\cdot)$ is continuous and nonnegative definite, then the conclusions of Theorem 5 hold. We summarize this as the following.

Theorem 6: Let $G(t)$ and $A(t)$ be as in Theorem 5. Let $X(t)=A(t)G(t)$, and let $g \in \mathcal{G}$. Then statements A and B of Theorem 5 hold.

Concluding Remarks

We have presented some results concerning the second moment properties of the output of a ZNL with a random input. It should be noted that the results we presented in Theorems 3-6 cannot be extended to arbitrary non-Gaussian random processes. For instance, it is very easy to exhibit a non-Gaussian random process which is not strictly bandlimited, but is such that certain zero memory nonlinear transformations upon it result in a strictly bandlimited random process.

Acknowledgement

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